CHAPTER 26

Monotonic Functions and Applications of Derivatives

Exercise

1. For the curve $y = xe^x$, the point
   (a) $x = -1$ is a point of minima
   (b) $x = 0$ is a point of minima
   (c) $x = -1$ is a point of maxima
   (d) $x = 0$ is a point of maxima
2. $f(x) = \sin^4 x + \cos^4 x$ increasing in
   (a) $\left]0, \frac{\pi}{8}\right[$ (b) $\left]\frac{\pi}{4}, \frac{\pi}{2}\right[$
   (c) $\left]\frac{3\pi}{8}, \frac{5\pi}{8}\right[$ (d) $\left]\frac{5\pi}{8}, \frac{3\pi}{8}\right[$
3. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the X-axis, is
   (a) $y = 0$ (b) $y = 1$
   (c) $y = 2$ (d) $y = 3$
4. If $f(x) = \frac{a\sin x + 2\cos x}{\sin x + \cos x}$ is increasing for all values of x, then
   (a) $a < 1$ (b) $a < 2$
   (c) $a > 1$ (d) $a > 2$
5. $f(x) = x^3 - 3x^2 - 105x + 25$ is decreasing in the interval
   (a) $]-\infty, -7[$ (b) $]-5, -\infty[$
   (c) $]-7, 5[$ (d) None of these
6. The maximum value of $(1/x)^x$ is
   (a) e (b) $e^{1/e}$
   (c) $(1/e)^e$ (d) 1
7. A wire of length 40 cm is to be bent into a rectangle. For the area of rectangle to be largest
   (a) one side of rectangle should be twice the other
   (b) one side should be 2.5 times the other
   (c) one side should be 1.5 times the other
   (d) the two sides should be equal
8. The interval in which $f(x) = (x + 3)^3 (x - 1)^3$ is increasing, is
   (a) $]-\infty, 1[$ (b) $]-1, 3[$
   (c) $]3, \infty[$ (d) $[-1, \infty[$
9. $f(x) = (x^5 - 5x^4 + 5x^3 - 1)$ has
   (a) maxima at $x = 3$, minima at $x = 0$ and neither at $x = 1$
   (b) maxima at $x = 0$, minima at $x = 3$ and neither at $x = 1$
   (c) maxima at $x = 3$, minima at $x = 1$ and neither at $x = 0$
   (d) maxima at $x = 1$, minima at $x = 3$ and neither at $x = 0$
10. $f(x) = \{x(x - 3)\}^2$ is increasing in
    (a) $]0, \infty[$ (b) $]-\infty, 0[$
    (c) $]1, 3[$ (d) $\left[0, \frac{3}{2}\right] \cup [3, \infty[$
11. If $a < 0$, the function $f(x) = (e^{ax} + e^{-ax})$ is monotonically decreasing for values of x, given by
    (a) $x > 0$ (b) $x < 0$
    (c) $x > 1$ (d) $x < 1$
12. The function $f(x) = \cos x - 2px$ is monotonically decreasing for
    (a) $p < \frac{1}{2}$ (b) $p > \frac{1}{2}$
    (c) $p < 2$ (d) $p > 2$
13. The real number x when added to its inverse gives the minimum value of the sum at x equal to
    (a) -2 (b) -1
    (c) 1 (d) 2
14. On the interval [0, 1] the function $x^{25}(1 - x)^{75}$ takes the maximum value at the point
    (a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) 0

15. All the values of λ for which $f(x) = \lambda x^3 - 9x^2 + 9x + 10$ is an increasing function on R, are given by
(a) $\lambda > 3$ (b) $\lambda < 3$
(c) $\lambda \geq 3$ (d) None of these

16. The value of k in order that
$f(x) = \sin x - \cos x - kx + b$
decreases for all real values of x, is given by
(a) $k < 1$ (b) $k > 1$
(c) $k > \sqrt{2}$ (d) $k < \sqrt{2}$

17. The fuel charges for running a train are proportional to the square of the speed generated in miles per hour and costs ₹ 48 per hour at 16 miles per hour. The most economical speed if the fixed charges, *i.e.*, salaries, *etc.*, amount to ₹ 300 per hour is
(a) 10 (b) 20
(c) 30 (d) 40

18. The function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1 = 0$ has a local maximum at $x = \alpha$ and a local minimum at $x = \beta$ such that $\beta = \alpha^2$ then a is equal to
(a) 0 (b) 1/4
(c) 2 (d) either 0 or 2

19. Given that $f(x) = x^{1/x}$, $x > 0$ has the maximum value at $x = e$, then
(a) $e^\pi > \pi^e$ (b) $e^\pi < \pi^e$
(c) $e^\pi = \pi^e$ (d) $e^\pi \leq \pi^e$

20. The minimum value of $ax + by$, where $xy = p^2$ is
(a) $2p\sqrt{ab}$ (b) $2ab\sqrt{p}$
(c) $-2p\sqrt{ab}$ (d) None of these

21. If a function $f(x)$ has $f'(a) = 0$ and $f''(a) = 0$, then
(a) $x = a$ is a maximum for $f(x)$.
(b) $x = a$ is a minimum for $f(x)$.
(c) it is difficult to imply (a) and (b).
(d) $f(x)$ is necessarily a constant function.

22. The norm al to the curve $x = a(1 + \cos\theta)$, $y = a \sin\theta$ at 'θ' always passes through the fixed point
(a) (a, a) (b) $(a, 0)$
(c) $(0, a)$ (d) None of these

23. The values of a for which $y = x^2 + ax + 25$ touches the axis of X are
(a) ± 5 (b) ± 10
(c) ± 15 (d) None of these

24. The equation of the tangent at the point t on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is
(a) $y = (x - at) \cdot \tan\left(\frac{t}{2}\right)$
(b) $y = (x + at) \cdot \tan\left(\frac{t}{2}\right)$
(c) $y = (x - at) \cdot \cot\left(\frac{t}{2}\right)$
(d) None of these

25. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then
(a) $a > 0, b > 0$
(b) $a > 0, b < 0$
(c) $a < 0, b > 0$
(d) Both (b) and (c)

26. If $y = a \log|x| + bx^2 + x$ has its extreme values at $x = -1$ and $x = 2$, then
(a) $a = 2, b = -1$ (b) $a = 2, b = -\frac{1}{2}$
(c) $a = -2, b = \frac{1}{2}$ (d) None of these

27. If the tangent at any point on the curve $y = x^3 - \lambda x^2 + x + 1$ makes an acute angle with the (+)ve direction of X-axis, then
(a) $\lambda > 0$ (b) $\lambda \leq \sqrt{3}$
(c) $-\sqrt{3} \leq \lambda \leq \sqrt{3}$ (d) None of these

28. Equation of the normal to the curve
$y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$ is
(a) $x = \pi$ (b) $x = 2$
(c) $x + \pi = 0$ (d) $x = \frac{\pi}{2}$

29. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is
(a) $\frac{22}{7}$ (b) $\frac{6}{7}$
(c) -6 (d) None of these

30. The normal to the curve $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any θ is such that
(a) it makes a constant angle with X-axis
(b) it passes through the origin
(c) it is at a constant distance from the origin
(d) None of the above

ANSWERS

1.	(a)	**2.**	(b)	**3.**	(d)	**4.**	(d)	**5.**	(d)	**6.**	(b)	**7.**	(d)	**8.**	(d)	**9.**	(d)	**10.**	(d)
11.	(b)	**12.**	(b)	**13.**	(c)	**14.**	(c)	**15.**	(c)	**16.**	(c)	**17.**	(d)	**18.**	(c)	**19.**	(a)	**20.**	(a)
21.	(c)	**22.**	(b)	**23.**	(b)	**24.**	(a)	**25.**	(d)	**26.**	(b)	**27.**	(c)	**28.**	(d)	**29.**	(b)	**30.**	(c)

Explanations

1. (a) $y = xe^x \Rightarrow \dfrac{dy}{dx} = 0$

$\Rightarrow e^x(x + 1) = 0 \Rightarrow x = -1$

$\Rightarrow \dfrac{d^2y}{dx^2} = e^x(x+2)$

At $x = -1, \dfrac{d^2y}{dx^2} > 0$

So, y is minimum at $x = -1$ and $x = -1$ is a point of minima.

2. (b) $f(x) = \sin^4 x + \cos^4 x$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cdot \cos^2 x$

$= 1 - \dfrac{1}{2}(\sin 2x)^2$

$\Rightarrow f'(x) = \dfrac{1}{2} \times 4 \sin 2x \cos 2x = -\sin 4x$

$\Rightarrow f'(x) > 0 \Rightarrow \sin 4x < 0$

$\Rightarrow x < 4x < 2\pi \Rightarrow \dfrac{\pi}{4} < x < \dfrac{\pi}{2}$

$\Rightarrow x \in]\dfrac{\pi}{4}, \dfrac{\pi}{2}[$

3. (d) $y = x + \dfrac{4}{x^2} \Rightarrow \dfrac{dy}{dx} = 1 - \dfrac{8}{x^3}$

Slope of X-axis $= 0$

$\Rightarrow 1 - \dfrac{8}{x^3} = 0 \Rightarrow x^3 = 8$ or $x = 2$

So, $y = 2 + \dfrac{4}{4} = 3$

4. (d) $f(x) = \dfrac{a\sin x + 2\cos x}{\sin x + \cos x}$

$\Rightarrow f'(x) = \dfrac{a-2}{(\sin x + \cos x)^2}$

$\therefore f'(x) > 0 \Leftrightarrow a - 2 > 0$

$\Rightarrow a > 2$

5. (d) $f(x) = x^3 - 3x^2 - 105x + 25$

$f'(x) = 3x^2 - 6x - 105 = 3(x + 5)(x - 7)$

For decreasing function $f'(x) < 0$

$\Rightarrow 3(x + 5)(x - 7) < 0$

$\Rightarrow x \in]-5, 7[$

6. (b) $y = \left(\dfrac{1}{x}\right)^x = x^{-x} \Rightarrow \log y = -x \log x$

$\dfrac{dy}{dx} = -x^{-x}(1 + \log x) = 0$

$\Rightarrow x = \dfrac{1}{e}$

$\Rightarrow \dfrac{d^2y}{dx^2} = x^{-x}\left\{\dfrac{-1}{x} + (1 + \log x)^2\right\}$

$\Rightarrow \left(\dfrac{d^2y}{dx^2}\right)_{(x=1/e)} < 0$

So, $x = \dfrac{1}{e}$ is a point of maxima.

Hence, maximum value $= e^{1/e}$

7. (d) $l + b = 40$ (Given)

$\because A = lb = l(40 - l) = 40l - l^2$

$\Rightarrow \dfrac{dA}{dl} = 40 - 2l$

Put $\dfrac{dA}{dl} = 0 \Rightarrow 40 - 2l = 0$

$\Rightarrow l = 20$

So, area is maximum at $l = 20$

Hence, both sides are equal.

8. (d) $f(x) = (x + 3)^3(x - 1)^3$

$\Rightarrow f'(x) = 3(x + 3)^2(x - 1)^3 + 3(x - 1)^2(x + 3)^3$

$= 6(x - 1)^2(x + 3)^2(x + 1)$

$\because f(x)$ is increasing functions, so, $f'(x) \geq 0$

$\Rightarrow (x - 1)^2(x + 3)^2(x + 1) \geq 0$

$\Rightarrow x \geq -1 \Rightarrow x \in [-1\ \infty[$

9. (d) $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$= 5x^2(x^2 - 4x + 3)$

$f''(x) = 10x(2x^2 - 6x + 3)$

$f'''(x) = 60x - 120x + 30$

Now, $f'(x) = 0$

$\Rightarrow 5x^2(x - 3)(x - 1) = 0 \Rightarrow x = 0, 1, 3.$

$f''(3) = 90 > 0$

$f''(1) = -10 < 0$

$f''(0) = 0$ and $f'''(0) = 30 \neq 0$

$\therefore f(x)$ has a maxima at $x = 1$, minima at $x = 3$ and neither at $x = 0$.

10. (d) $f(x) = [x(x - 3)]^2$

$\Rightarrow f'(x) = 2x(x - 3)(2x - 3)$

$\because f'(x) \geq 0$ {increasing function}

$\Rightarrow x(x - 3)(2x - 3) \geq 0$

$\Rightarrow x \in \left[0\ \dfrac{3}{2}\right] \cup [3\ \infty[$

11. (b) $\because a < 0$, let $a = -p$, where $p > 0$

$f(x) = e^{-px} + e^{px}, p > 0$

$\Rightarrow f'(x) = pe^{px} - pe^{-px} = p[e^{px} - e^{-px}]$

For montonically decreasing function

$f'(x) < 0 \Leftrightarrow e^{px} - e^{-px} < 0$

$\Leftrightarrow e^{px} < e^{-px}$

$\Leftrightarrow e^{2px} < 1 \Leftrightarrow x < 0$

$\therefore f(x)$ is monotonically decreasing for $x < 0$.

12. (b) $f(x) = \cos x - 2px$

$\Rightarrow f'(x) = -\sin x - 2p = -(\sin x + 2p)$

$\Rightarrow$ For monotonically decreasing function

$f'(x) < 0 \Rightarrow \sin x + 2p > 0$

$\Rightarrow \frac{1}{2}\sin x + p > 0$

$p > -\frac{1}{2}\sin x$

$\because -1 \le \sin x \le 1$

$\therefore -\frac{1}{2} \le \frac{1}{2}\sin x \le \frac{1}{2}$

So, $p > \frac{1}{2}$

13. (c) Let $A = x + \frac{1}{x}$

$\frac{dA}{dx} = 0$

$\Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = 1, -1$

$\Rightarrow \frac{d^2A}{dx^2} = \frac{2}{x^3}$

$\frac{d^2A}{dx^2} > 0$ at $x = 1$

So, A is minimum at $x = 1$.

14. (c) Let $y = x^{25}(1-x)^{75}$

$\frac{dy}{dx} = 0$

$-x^{25} \cdot 75(1-x)^{74} + 25x^{24}(1-x)^{75} = 0$

$25x^{24}(1-x)^{74}[1 - x - 3x] = 0$

$\Rightarrow x = 0, 1, \frac{1}{4}$

At $x = 0$ and 1, $\frac{dy}{dx}$ does not change its sign.

But at $x = \frac{1}{4}$, $\frac{dy}{dx}$ changes its sign from (+) to (–)

So, point of maxima = 1/4

15. (c) $f(x) = \lambda x^3 - 9x^2 + 9x + 10$

For increasing function $f'(x) > 0$

$3\lambda x^2 - 18x + 9 > 0$

For this condition $\lambda > 0$ …(i)

and $D \le 0$ or $324 - 108\lambda \le 0$

$\Rightarrow \lambda \ge 3$ …(ii)

From eqs. (i) and (ii), $\lambda \ge 3$

16. (c) $f(x) = \sin x - \cos x - kx + b$

$f'(x) = \cos x + \sin x - k$

$= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) - k$

Max value of $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = \sqrt{2}$

$\because$ For $k > \sqrt{2}$, $f'(x) < 0 \ \forall\ x$

$\Rightarrow f$ is a decreasing function.

17. (d) Let the speed of train = v

and distance to be covered = s

So, total time taken = $\frac{s}{v}$ hours.

Given, cost of fuel per hour = kv^2 {k is constant.}

By given condition, $48 = k \cdot 16^2$

$\therefore k = \frac{3}{16}$

$\therefore$ Cost of fuel per hour = $\frac{3}{16}v^2$

Other charges per hour are ₹ 300.

So, charges per hour = $\frac{3}{16}v^2 + 300$

$\therefore$ Total expenses for the journey

$$E = \left(\frac{3}{16}v^2 + 300\right)\frac{s}{v} = s\left(\frac{3}{16}v + \frac{300}{v}\right)$$

$$\frac{dE}{dV} = s\left(\frac{3}{16} - \frac{300}{v^2}\right) = 0$$

$\Rightarrow v^2 = 1600 \Rightarrow v = 40$

$\frac{d^2E}{dS^2} = S\left(\frac{1600}{v^3}\right)$ = (+)ve for $v = 40$

$\Rightarrow E$ is minimum at $v = 40$

Hence, most economical speed = 40 miles/hour.

18. (c) $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

Put $f'(x) = 0$

$\Rightarrow 6x^2 - 18ax + 12a^2 = 0$

or $6(x - a)(x - 2a) = 0$

or $x = a, x = 2a$

$f''(x) = 12x - 18a$

$f''(a) = -6a$

$f''(2a) = 6a$

So, $f(x)$ is minimum at $\beta = 2a$ and maximum at $\alpha = 2$

Then, $\beta = \alpha^2$

$\Rightarrow 2a = a^2$ or $a = 2$

19. (a) $f(x) = x^{1/x}, x > 0$

Given, $x = e$ is a point of maxima.

Hence, $f(e) > f(x) \ \forall\ x > 0$

or $f(e) > f(\pi)$

$\Rightarrow e^{1/e} > \pi^{1/\pi} \Rightarrow e^{\pi} > \pi^{e}$

20. (a) $z = ax + by$, where $xy = p^2$

$\Rightarrow z = ax + \frac{bp^2}{x}$

$\Rightarrow \frac{dz}{dx} = a - \frac{bp^2}{x^2} = 0 \Rightarrow x = p\sqrt{\frac{b}{a}}$

$\therefore \quad \dfrac{d^2z}{dx^2} = \dfrac{2bp^2}{x^3}$

$\because$ x is a point of minimum.

Minimum value of $z = ap\sqrt{\dfrac{b}{a}} + \dfrac{bp^2}{p}\sqrt{\dfrac{a}{b}} = 2p\sqrt{ab}$

21. (c) For max. or min. $f''(a)$ should be (–)ve or (+)ve value respectively.
Hence, it is difficult to imply option (a) and (b).

22. (b) $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\cos\theta}{a-(\sin\theta)} - \cot\theta$

Equation of normal

$y - a\sin\theta = \dfrac{1}{-\cot\theta}\{x - a(1+\cos\theta)\}$

$y - a\sin\theta = \dfrac{\sin\theta}{\cos\theta}\{x - a(1+\cos\theta)\}$

$x\sin\theta - y\cos\theta = a\sin\theta$

If is satisfied by $(a, 0)$. So, passes through $(a, 0)$.

23. (b) $y = x^2 + ax + 25$ …(i)

$\dfrac{dy}{dx} = 2x + a$

Let it touches the X-axis at (x_1, y_1), then slope at $(x_1, y_1) = 2x_1 + a$

Slope of X-axis = 0

$\Rightarrow 2x_1 + a = 0$ or $x_1 = -\dfrac{a}{2}$

and $y_1 = \dfrac{a^2}{4} - \dfrac{a^2}{2} + 25 = -\dfrac{a^2}{4} + 25$

But it lies on X-axis, so $-\dfrac{a^2}{4} + 25 = 0$

or $a = \pm 10$

24. (a) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a\sin t}{a(1+\cos t)} = \tan\dfrac{t}{2}$

$\Rightarrow y - a(1 - \cos t) = \tan\dfrac{t}{2}[x - a(t + \sin t)]$

or $y - a\left(2\sin^2\dfrac{t}{2}\right) = (x - at)\tan\dfrac{t}{2} - 2a\sin^2\dfrac{t}{2}$

or $y = (x - at)\tan\dfrac{t}{2}$

25. (d) $y = \dfrac{1}{x}$

$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$; slope of normal $= x^2$

Slope of line $ax + by + c = 0$ is $-\dfrac{a}{b}$.

$\Rightarrow x^2 = -\dfrac{a}{b}$

or $\dfrac{a}{b}$ is (–)ve.

$\Rightarrow a > 0, b < 0$ or $a < 0, b > 0$

26. (b) $y = a\log|x| + bx^2 + x$

Put $\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{a}{x} + 2bx + 1 = 0$ at $x = -1$ and 2

So, $a + 2b = 1$ and $a + 8b = -2$

$\Rightarrow a = 2$ and $b = -\dfrac{1}{2}$

27. (c) $y = x^3 - \lambda x^2 + x + 1$

$\dfrac{dy}{dx} = 3x^2 - 2\lambda x + 1$

Tangent makes acute angle with positive X-axis.

So, $\dfrac{dy}{dx} \geq 0$

$3x^2 - 2\lambda x + 1 \geq 0$

$\Rightarrow 4\lambda^2 - 4(3) \leq 0$

$(\lambda - \sqrt{3})(\lambda + \sqrt{3}) \leq 0$

$\Rightarrow -\sqrt{3} \leq \lambda \leq \sqrt{3}$

28. (d) $y = x + \sin x\cos x$

$\dfrac{dy}{dx} = 1 + \cos 2x$

At $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \dfrac{dy}{dx} = 0$

So, slope of normal $= \infty$

$\therefore$ Equation of normal,

$\dfrac{y - \frac{\pi}{2}}{x - \frac{\pi}{2}} = m = \infty$

$\Rightarrow x = \dfrac{\pi}{2}$

29. (b) $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$

$t = 2$ for the point $(2, -1)$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{4t-2}{2t+3} = \dfrac{6}{7}$ for $t = 2$

30. (c) $y = a(\sin\theta - \theta\cos\theta), x = a(\cos\theta + \theta\sin\theta)$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\sin\theta}{\cos\theta}$ {slope of tangent}

$\therefore$ Slope of normal $= -\dfrac{\cos\theta}{\sin\theta}$

$\therefore$ Normal is $[y - (a\sin\theta - a\theta\cos\theta)] = \dfrac{\cos\theta}{\sin\theta}$
$[x - (a\cos\theta + a\theta\sin\theta)]$

or $y\sin\theta + x\cos\theta = a$

Its distance from the origin is

$\dfrac{a}{\sqrt{\cos^2\theta + \sin^2\theta}} = a$, *i.e.*, constant.